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On an asymptotic expansion for the von Mises ω^2 statistic

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In honour of Professor Károly Tandori on his 50th birthday

§ 1. Introduction. There are two classical types of statistics for testing the "goodness-of-fit" hypothesis that the distribution function of a statistical population coincides with the fully determined continuous distribution function $F(x)$. The defining statistics for one of them are those of Kolmogorov's $\sup |F_n(x)-F(x)|$ and Smirnov's $\sup (F_n(x)-F(x))$, while for the other $\omega_n^2=n \int\limits_{-\infty}^{\infty} (F_n(x)-F(x))^2 dF(x)$ of von Mises, where $F_n(x)$ denotes the empirical distribution function based on a random sample of size n. Considering the latter one, CRAMÉR [4] was the first in 1928, who proposed a statistic similar to ω_n^2, while ω_n^2 itself was proposed by von MISES [20] in 1931. He proved that for any complex λ

$$\text{(1)} \qquad \lim_{n\to\infty} \mathbf{E} e^{-\lambda\omega_n^2} = \prod_{k=1}^{\infty} \left(1+\frac{2\lambda}{k^2\pi^2}\right)^{-1/2} = \left(\frac{\sqrt{2\lambda}}{\sinh \sqrt{2\lambda}}\right)^{1/2},$$

provided the null hypothesis holds true, and this we will assume throughout. The limiting Laplace—Stieltjes transform was first inverted by SMIRNOV [32] in 1937, who, in this way, proved the following limit distribution theorem

$$\text{(2)} \qquad \lim_{n\to\infty} \mathbf{P}\{\omega_n^2 < x\} = \lim_{n\to\infty} V_n(x) = V(x),$$

where

$$V(x) = 1-\frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\int\limits_{(2k-1)\pi}^{2k\pi}(-u\sin u)^{-1/2}e^{-u^2x/2}du.$$

Another expression for $V(x)$, due to ANDERSON and DARLING [1] dates back to 1952:

$$V(x) = \frac{1}{\pi\sqrt{x}}\sum_{k=0}^{\infty}(-1)^k\binom{-1/2}{k}\sqrt{4k+1}\, e^{-\frac{(4k+1)^2}{16x}} B_{1/4}\left(\frac{(4k+1)^2}{16x}\right),$$

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where $B_{1/4}(y)$ is a standard Bessel function. As ω_n^2 (just like the Kolmogorov—Smirnov statistics) is distribution-free, it will not be a loss of generality to assume that the underlying population is uniformly distributed on the interval [0, 1] $\big(F(x)=x$ for $x\in[0, 1]\big)$, when investigating the asymptotic behavior of its distribution. Then, introducing the empirical process $Y_n(t)=\sqrt{n}\big(F_n(t)-t\big)$, $0\leqq t\leqq 1$, we have $\omega_n^2=\int\limits_0^1 Y_n^2(t)\,dt$. $Y_n(t)$ is a random element of Skorohod's space of functions on [0, 1], $D[0, 1]$, having discontinuities only of the first kind. It is known that Y_n converges weakly to the Brownian Bridge $B(t)$, a Gaussian process on [0, 1] with expectation 0 and covariance function $s(1-t)$ for $0\leqq s\leqq t\leqq 1$ (see [12], [13], [3] or [6] in these *Acta*). Introducing $\omega^2=\int\limits_0^1 B^2(t)\,dt$, this latter result of Doob and Donsker immediately gives

$$V(x)=\mathbf{P}\{\omega^2<x\},\quad \mathbf{E}e^{-\lambda\omega^2}=\left(\frac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}\right)^{1/2},$$

since the square-integral is a continuous functional in the topology of $D[0, 1]$.

In the Kolmogorov—Smirnov case not only the exact rates of convergence $\left[O\left(\frac{1}{\sqrt{n}}\right)\right]$ for the appropriate limit distribution relations are known, but asymptotic expansions are also available from the first half of the 50's (see [14]). At the same time, such a complete set of results seemed to be far in the von Mises case; this was also emphasized by Durbin and Bickel in their recent survey papers [11] and [2]. This kind of asymptotic behavior of $V_n(x)=\mathbf{P}\{\omega_n^2<x\}$ is all the more important since, next to nothing is known about the exact distribution of ω_n^2. With the exception of the exact formulae for $n=1, 2, 3$ in [19] only an extreme lower tail of the distribution ([33]), the exponential decrease of the upper tail ([26]) and the first four moments of ω_n^2 ([24]) are known for any further n.

Put $\Delta_n=\sup\limits_{-\infty<x<\infty}|V_n(x)-V(x)|$. The first estimate was given by Kandelaki [15] in 1965, namely that $\Delta_n\leqq C(\log n)^{-1/4}$, with some absolute constant C. It was expected that Δ_n should be estimable the following way: for any $\varepsilon>0$ there should exist a constant $b(\varepsilon)$ such that for each n

$$\Delta_n\leqq b(\varepsilon)\frac{n^{\varepsilon}}{n^{a}}, \tag{3}$$

with some $a>0$. Sazonov first proved (3) with $a=\frac{1}{10}$ ([28]) and then with $a=\frac{1}{6}$ ([29]). Using a Skorohod embedding (see [31]) Rosenkrantz [27] concluded in $\Delta_n=O\big((\log n)^{3/2}n^{-1/5}\big)$, which is, of course, better than (3) with $a=\frac{1}{5}$. Next, by the same embedding KIEFER [17] proved $\Delta_n = O((\log n)^{3/2} n^{-1/4})$ and, independently, NIKITIN [21] announced $\Delta_n = O((\log n)^{5/4} n^{-1/4})$. Kiefer also proved (see also SAWYER [30]) that the Skorohod embedding cannot give more than $n^{1/4}$ in the denominator. Later ORLOV [22] increased a in (3) to $\frac{1}{3}$. Finally, in a new long paper [23] ORLOV proved that (3) does not hold with $a > 1$ and holds with $a = \frac{1}{2}$.

In § 2 of the present paper a refinement of Orlov's estimate is given which turns out to be the best rate that can be achieved by all the previously existing methods. In § 3 a complete asymptotic expansion for the Laplace transform of ω_n^2 is given. (In this connection we have to mention an early result of DARLING [9], which he announced, without proof, in 1960. This is a one-term expansion for $\mathbf{E}e^{-\lambda\omega_n^2}$, but only for real positive λ, and so there is no hope to invert it). The first outline of its proof (without the estimation of the dependence on λ of the remainder term) was published in [7] and its details in [8]. The treatment of dependence on λ is new here. In § 4 the problem of inversion of this expansion is treated, without reaching the final answer. A few, not completely rigorous throughts on this inversion were also included in [8]. § 5 tries to motivate our conjecture concerning the final form the asymptotic expansion for $V_n(x)$ and the exact rate of convergence of Δ_n.

§ 2. A rate of convergence. The following theorem is true.

Theorem 1. $\Delta_n = \sup\limits_{-\infty<x<\infty} |V_n(x) - V(x)| = O\left(\frac{\log n}{\sqrt{n}}\right)$.

The proof is entirely based on one of the recent and very important results of Komlós, Major and Tusnády.

Theorem A. (KOMLÓS, MAJOR and TUSNÁDI [18]) *For each n there exists an empirical distribution function $\tilde{F}_n(t)$ of independent, uniformly distributed random variables on* [0, 1] *and a Brownian Bridge $B_n(t)$ such that for the empirical process $\tilde{Y}_n(t) = \sqrt{n}\,(\tilde{F}_n(t) - t)$ we have, for each x*

$$\mathbf{P}\left\{\sup_{0\le t\le 1} |\tilde{Y}_n(t) - B_n(t)| > \frac{A\log n + x}{\sqrt{n}}\right\} < Be^{-Cx},$$

where A, B and C are positive absolute constants. Putting $x = K\log n$ so that $KC > 1$ and using the Borel—Cantelli lemma one gets

$$\sup_{0\le t\le 1} |\tilde{Y}_n(t) - B_n(t)| = O\left(\frac{\log n}{\sqrt{n}}\right),$$

with probability 1.

Proof of theorem 1. Suppose that the random variables

$$\tilde{\omega}_n^2 = \int_0^1 \tilde{Y}_n^2(t)\,dt \quad \text{and} \quad \tilde{\omega}^2 = \int_0^1 B_n^2(t)\,dt$$

are built on the processes $\tilde{Y}_n(t)$ and $B_n(t)$ of Theorem A. Naturally, their distribution functions are $V_n(x)$ and $V(x)$, respectively. Then, by Theorem A, there exists with probability 1 a constant K such that

$$|\tilde{\omega}_n^2-\tilde{\omega}^2| = \left|\int_0^1 (\tilde{Y}_n(t)-B_n(t))(\tilde{Y}_n(t)+B_n(t))\,dt\right| \leqq K\frac{\log n}{\sqrt{n}}\int_0^1 |\tilde{Y}_n(t)+B_n(t)|\,dt \leqq$$

$$\leqq K\frac{\log n}{\sqrt{n}}\left(\int_0^1 |\tilde{Y}_n(t)-B_n(t)|\,dt+2\int_0^1 |B_n(t)|\,dt\right) \leqq K^2\frac{\log^2 n}{n}+2K\frac{\log n}{\sqrt{n}}\,\tilde{\omega},$$

where the last inequality follows from that of Buniakovsky—Schwarz. That is we have

$$\mathbf{P}\left\{|\tilde{\omega}_n^2-\tilde{\omega}^2| > \frac{1}{4}\varepsilon_n^2+\varepsilon_n\tilde{\omega}\right\} = 0, \tag{4}$$

where $\varepsilon_n=2K\dfrac{\log n}{\sqrt{n}}$. Solving the corresponding quadratic inequalities for the sets $A_n=\left\{\tilde{\omega}^2<x-\dfrac{1}{4}\varepsilon_n^2-\varepsilon_n\tilde{\omega}\right\}$ and $B_n=\left\{\tilde{\omega}^2<x+\dfrac{1}{4}\varepsilon_n^2+\varepsilon_n\tilde{\omega}\right\}$ we find that

$$A_n = \left\{\tilde{\omega}^2 < x+\frac{1}{4}\varepsilon_n^2-\sqrt{\varepsilon_n^2 x}\right\} \quad \text{and} \quad B_n = \left\{\tilde{\omega}^2 < x+\frac{3}{4}\varepsilon_n^2+\left(\frac{1}{2}\varepsilon_n^4+\varepsilon_n^2 x\right)^{1/2}\right\}.$$

Consequently, from (4) one gets

$$V\left(x+\frac{1}{4}\varepsilon_n^2-\sqrt{\varepsilon_n^2 x}\right) \leqq V_n(x) \leqq V\left(x+\frac{3}{4}\varepsilon_n^2+\left(\frac{1}{2}\varepsilon_n^4+\varepsilon_n^2 x\right)^{1/2}\right),$$

and, a fortiori,

$$V\left(x-\frac{3}{4}\varepsilon_n^2-\left(\frac{1}{2}\varepsilon_n^4+\varepsilon_n^2 x\right)^{1/2}\right) \leqq V_n(x) \leqq V\left(x+\frac{3}{4}\varepsilon_n^2+\left(\frac{1}{2}\varepsilon_n^4+\varepsilon_n^2 x\right)^{1/2}\right).$$

This, with some constants A, B, C and D, implies

$$|V_n(x)-V(x)| \leqq \mathbf{P}\left\{x-\frac{3}{4}\varepsilon_n^2-\varepsilon_n\left(\frac{1}{2}\varepsilon_n^2+x\right)^{1/2} \leqq \omega^2 \leqq x+\frac{3}{4}\varepsilon_n^2+\varepsilon_n\left(\frac{1}{2}\varepsilon_n^2+x\right)^{1/2}\right\} \leqq$$

$$\leqq v(x)\left(A\frac{\log^2 n}{n}+B\frac{\log n}{\sqrt{n}}\left(C\frac{\log^2 n}{n}+x\right)^{1/2}\right) \leqq A\frac{\log^2 n}{n}v(x)+D\frac{\log n}{\sqrt{n}}v(x)\sqrt{x},$$

where $v(x)=\dfrac{d}{dx}V(x)$ is the density function of ω^2. Later (Lemma 8 in § 5) we will see, that $v(x)$ as well as $\sqrt{x}\,v(x)$ are bounded functions on the whole line, and thus the theorem is proved.

It is worthwhile here to remark that all the previous methods for getting a rate of convergence for Δ_n (just to mention the characteristic ones of ROSENKRANTZ [27], ORLOV [23] and the proof of the above Theorem 1) are based on some kind of approximation of the empirical process. From the nearness of the latter approximation then resulted a nearness of $V_n(x)$ and $V(x)$. Of course, the applied method in the proof of Theorem 1, i.e. the use of the $O\left(\frac{\log n}{\sqrt{n}}\right)$ approximation of Komlós, Major and Tusnády cannot give a better rate for Δ_n than $O\left(\frac{\log n}{\sqrt{n}}\right)$. But at the same time the Brownian Bridge of Komlós, Major and Tusnády is the best approximation for the empirical process (see also in a forthcoming book [5] of M. CSÖRGŐ and P. RÉVÉSZ). Therefore the following conclusion is true: one cannot get a better rate of convergence for Δ_n than $O\left(\frac{\log n}{\sqrt{n}}\right)$ of Theorem 1 via first approximating the empirical process.

We remark also that our rate was thought to be desirable (if not the best) by ROSENKRANTZ [27] and later by KIEFER [17] and BICKEL [2]. On the grounds of the following two paragraphs, however, one can even expect more, namely, that Δ_n has the order of $\frac{1}{n}$.

§ 3. Asymptotic expansion for the Laplace transform. In this section we prove:

Theorem 2. *For any complex λ, with $\operatorname{Re}\lambda\geqq 0$, natural s and real ε with $\varepsilon>0$,*

$$\mathbf{E}e^{-\lambda\omega_n^2}-\mathbf{E}e^{-\lambda\omega^2}=\sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k a_k(\lambda)+h_s(\lambda)\,O(n^{\varepsilon-(s+1)/2}),$$

where

$$a_k(\lambda)=\sum_{(i_1,\ldots,i_{2k})}{}' b_{i_1,\ldots,i_{2k}}\lambda^{k+H_{2k}}\mathbf{E}\exp\left\{-\lambda\int_0^1\alpha^2(t)\,dt\right\}\Pi_{i_1,\ldots,i_{2k}}.$$

Here $H_n=\sum_{j=1}^n i_j$, $\alpha(t)=W(t)-\int_0^1 W(x)dx$ and $W(t)$ is the standard Brownian Motion. Summation $\sum'$ is taken over all non-negative integer solutions $(i_1,\ldots,i_{2k})$ of the equation $i_1+2i_2+\ldots+2ki_{2k}=2k$. Further,

$$b_{i_1,\ldots,i_{2k}}=\frac{1}{i_1!\ldots i_{2k}}(-2)^{k+H_{2k}}$$

and

$$\Pi_{i_1,\ldots,i_{2k}}=\prod_{m=1}^{2k}\left\{\sum_{l=1}^{[(m+2)/2]}(-1)^{l-1}(l-1)!\sum_{(j_2,\ldots,j_s)}{}'' d_{j_2,\ldots,j_s}\prod_{r=2}^{s}\left(\int_0^1\alpha^r(t)\,dt\right)^{j_r}\right\}^{i_m},$$

where

$$d_{j_2,\ldots,j_s} = 1/(j_2!\ldots j_s!(2!)^{j_2}\ldots(s!)^{j_s})$$

and summation $\sum''$ *is taken over all non-negative integer solutions* $(j_2, \ldots, j_s)$ *of the equations* $j_2+\ldots+j_s=l$ *and* $2j_2+\ldots+sj_s=m+2$. $O(n^{-(s+1)/2+\varepsilon})$ *does not depend on* λ *any more, and for the function* $h_s(\lambda)$ *the following estimate is valid:*

$$|h_s(\lambda)| \leqq |\lambda|^{(s+2)(s+4)/2} \text{ if } |\lambda| > 1 \; \textit{and} \quad |h_s(\lambda)| \leqq |\lambda|^{1/2} \text{ if } |\lambda| \leqq 1.$$

Proof. Let the standard Wiener process $W(t)$ be independent of the empirical process $Y_n(t)$ (which is based on uniform [0,1] random variables $U_1, \ldots, U_n$) for each n, and let $g_n(x)$ be a (nonrandom) sample function of $Y_n(t)$. The random variable $\int_0^1 g_n(x)\,dW(x)$ is normal with mean 0 and variance $\int_0^1 g_n^2(x)\,dx$ (see e.g. [31]). Therefore

$$\mathbf{E}\exp\left\{\sqrt{-2\lambda}\int_0^1 g_n(x)\,dW(x)\right\} = \exp\left\{-\lambda\int_0^1 g_n^2(x)\,dx\right\},$$

whence

$$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \mathbf{E}\exp\left\{-\lambda\int_0^1 Y_n^2(x)\,dx\right\} = \mathbf{E}\exp\left\{\sqrt{-2\lambda}\int_0^1 Y_n(x)\,dW(x)\right\}.$$

If $g(x)$ is a continuous function on [0,1], then

$$\mathbf{E}\exp\left\{\sqrt{-2\lambda}\int_0^1 Y_n(x)\,dg(x)\right\} = \mathbf{E}\exp\left\{-\sqrt{-2\lambda}\int_0^1 g(x)\,dY_n(x)\right\} =$$

$$= \mathbf{E}\exp\left\{-\sqrt{-2\lambda/n}\sum_{k=1}^n g(U_k) + \sqrt{-2\lambda}\sqrt{n}\int_0^1 g(x)\,dx\right\} =$$

$$= \left\{\int_0^1 \exp\left(-\sqrt{-2\lambda/n}\left[g(t)-\int_0^1 g(x)\,dx\right]\right)dt\right\}^n.$$

Hence

$$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \mathbf{E}\{1+\theta_n(\lambda)\}^n,$$

where

$$\theta_n(\lambda) = \int_0^1 \left(\exp\left\{-\sqrt{-2\lambda/n}\,\alpha(t)\right\}-1\right)dt.$$

Let a real ε be given with $0<\varepsilon<\dfrac{1}{6(s+1)}$, where s is an arbitrary natural number. If for a set B the indicator of B is denoted by χ_B then we have

$$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \mathbf{E}\exp\{-\lambda\omega_n^2\}(1-\chi_{A_n^\varepsilon}) + \mathbf{E}\exp\{-\lambda\omega_n^2\}\chi_{A_n^\varepsilon} \tag{5}$$

where

$$A_n^\varepsilon = \{\sup_{0\leqq t\leqq 1}|W(t)| \leqq n^\varepsilon\}.$$

For the first term in (5), using the Buniakovsky—Schwarz inequality and then an estimate (see e.g. in [10]) for the tail probability of a Brownian Motion, one gets

$$(6)\qquad \left|\mathbf{E}\exp\{-\lambda\omega_n^2\}(1-\chi_{A_n^\varepsilon})\right| \leqq (\mathbf{P}\{\bar{A}_n^\varepsilon\})^{1/2} \leqq \frac{\sqrt[4]{2/\pi}}{n^{\varepsilon/2}}\exp\{-n^{2\varepsilon}/4\}.$$

As, trivially,

$$\mathbf{P}\{|\theta_n(\lambda)| \geqq 1\} \leqq \mathbf{P}\left\{\exp\left(2|\sqrt{-2\lambda}|\,\frac{\sup|W(t)|}{\sqrt{n}}\right) \geqq 2\right\},$$

on the set A_n^ε we have

$$|\theta_n(\lambda)| \leqq K\sqrt{|\lambda|}\,n^{\varepsilon-1/2} < 1$$

with some constant K not depending on λ if n is large enough. For the same n (which we will take in the sequel as large as needed without any further mention of it) therefore

$$\left|\sum_{m=s+1}^{\infty}\frac{(-1)^{m+1}}{m}\theta_n^m(\lambda)\right| < K^{s+1}|\lambda|^{(s+1)/2}n^{(\varepsilon-1/2)(s+1)}.$$

It follows then

$$(7)\qquad \mathbf{E}e^{-\lambda\omega_n^2}\chi_{A_n^\varepsilon} = \mathbf{E}\exp\{n\log(1+\theta_n(\lambda))\}\chi_{A_n^\varepsilon} =$$

$$= \mathbf{E}\exp\left\{n\sum_{m=1}^{s}\frac{(-1)^{m+1}}{m}\theta_n^m(\lambda)\right\}\chi_{A_n^\varepsilon}[1+h_{s-2}^1(\lambda)O(n^{(\varepsilon-1/2)(s+1)}n)] =$$

$$= \mathbf{E}\exp\left\{n\sum_{m=1}^{s}\frac{(-1)^{m+1}}{m}\theta_n^m(\lambda)\right\}\chi_{A_n^\varepsilon}+h_{s-2}^1(\lambda)O(n^{(\varepsilon-1/2)(s-1)+2\varepsilon}),$$

where the function $h_{s-2}^1(\lambda)$ is such that

$$|h_{s-2}^1(\lambda)| \leqq |\lambda|^{(s+1)/2}.$$

At the last equality it was taken into account that the first term of the last row tends to $\mathbf{E}e^{-\lambda\omega^2}$ as $n\to\infty$, the absolute value of which is less than 1 by $\operatorname{Re}\lambda \geqq 0$. Now we compute the powers of $\theta_n(\lambda)$. Putting $\beta(t) = -\sqrt{-2\lambda}\,\alpha(t)$ (which is $-\sqrt{-2\lambda}\,O(n^\varepsilon)$ on A_n^ε) and using the MacLaurin formula and the fact that $\int_0^1\alpha(t)\,dt=0$ we find on A_n^ε that

$$(8)\qquad \theta_n(\lambda) = \sum_{j=2}^{s}\left(\frac{1}{\sqrt{n}}\right)^j\int_0^1\frac{\beta^j(t)}{j!}\,dt + h_{s-2}^2(\lambda)O(n^{(\varepsilon-1/2)(s+1)}),$$

where

$$|h_{s-2}^2(\lambda)| \leqq |\lambda|^{(s+2)/2} \text{ if } |\lambda| > 1, \quad\text{and}\quad |h_{s-2}^2(\lambda)| \leqq |\lambda|^{1/2} \text{ if } |\lambda| \leqq 1.$$

There, for the estimation of the remainder, the simple fact that for $j_1 \leqq j_2$

(9) $$|\lambda|^{k_1} O(n^{(\varepsilon-1/2)j_1}) + |\lambda|^{k_2} O(n^{(\varepsilon-1/2)j_2}) \leqq \begin{cases} |\lambda|^{\max(k_1,k_2)} O(n^{(\varepsilon-1/2)j_1}), & \text{if } |\lambda| > 1, \\ |\lambda|^{\min(k_1,k_2)} O(n^{(\varepsilon-1/2)j_1}), & \text{if } |\lambda| \leqq 1 \end{cases}$$

was used and will be often in the sequel. In what follows, all the figuring functions $h(\lambda)$ with different (lower and upper) indices are majorized in absolute value by $|\lambda|^{1/2}$ if $|\lambda| \leqq 1$ and if some assertion of the type of $|h(\lambda)| \leqq |\lambda|^r$ (with some r) appears, then it refers to the case $|\lambda| > 1$. Using (9) several times with $|h_j(\lambda)| \leqq |\lambda|^{j/2}$ we get on A_n^ε from (8)

(10) $$\theta_n^m(\lambda) = \left\{\sum_{j=2}^{s} \left(\frac{1}{\sqrt{n}}\right)^j \int_0^1 \frac{\beta^j(t)}{j!}\,dt\right\}^m +$$

$$+ \sum_{\substack{k_1+k_2=m \\ (k_1,k_2)\neq(m,0)}} \left(\sum_{j=2}^{s} h_j(\lambda) O(n^{(\varepsilon-1/2)j})\right)^{k_1} \left(h_{s-2}^2(\lambda) O(n^{(\varepsilon-1/2)(s+1)})\right)^{k_2} =$$

$$= \sum_{i_2+\ldots+i_s=m} \frac{m!}{i_2!\ldots i_s!} \left(\frac{1}{\sqrt{n}}\right)^{2i_2+\ldots+si_s} \prod_{k=2}^{s} \left(\int_0^1 \frac{\beta^k(t)}{k!}\,dt\right)^{i_k} + h_{s,m}(\lambda) O(n^{(\varepsilon-1/2)[s+1+2(m-1)]}),$$

where

$$|h_{s,m}(\lambda)| \leqq |\lambda|^{(s+2)m/2}.$$

Multiplying by n and writing out explicitly the first term of the m-summation, and using again (9), we have

(11) $$\mathbf{E}\exp\left\{n \sum_{m=1}^{s} \frac{(-1)^{m+1}}{m} \theta_n^m(\lambda)\right\} \chi_{A_n^\varepsilon} =$$

$$= \mathbf{E}\exp\left\{-\lambda \int_0^1 \alpha^2(t)\,dt + \sum_{j=1}^{s-2} \left(\frac{1}{\sqrt{n}}\right)^j \int_0^1 \frac{\beta^{j+2}(t)}{(j+2)!}\,dt +\right.$$

$$\left. + n \sum_{m=2}^{s} \frac{(-1)^{m+1}}{m} \sum_{j=2m}^{sm} \left(\frac{1}{\sqrt{n}}\right)^j (-\sqrt{-2\lambda})^j q_j^{(m)} + h_{s-2}^2(\lambda) O(n^{(\varepsilon-1/2)(s+1)+1})\right\} \chi_{A_n^\varepsilon} =$$

$$= \mathbf{E}\exp\left\{-\lambda \int_0^1 \alpha^2(t)\,dt + \sum_{m=1}^{s} \sum_{j=2m-2}^{sm-2} \left(\frac{1}{\sqrt{n}}\right)^j p_j^{(m)}\right\} \chi_{A_n^\varepsilon} + h_{s-2}^3(\lambda) O(n^{(\varepsilon-1/2)(s-1)+2\varepsilon}),$$

where

(12) $$q_j^{(m)} = \sum_{\substack{2i_2+\ldots+si_s=j \\ i_2+\ldots+i_s=m}}'' \frac{m!}{i_2!\ldots i_s!} \prod_{k=2}^{s} \left(\int_0^1 \frac{\alpha^k(t)}{k!}\,dt\right)^{i_k}$$

and $p_0^{(1)} = 0$, while for $j = 1, \ldots, s^2-2$ and $m = 1, \ldots, s$

(13) $$p_j^{(m)} = p_j^{(m)}(\lambda) = \frac{(-1)^{m+1}}{m} (-\sqrt{-2\lambda})^{j+2} q_{j+2}^{(m)},$$

furthermore

$$|h_{s-2}^3(\lambda)| \leqq |\lambda|^{(s+2)s/2}.$$

Now we break up the double-sum in the exponent of the above expected value according to powers of $\frac{1}{\sqrt{n}}$ and estimate all the terms of this sum on A_n^ε where the power of $\frac{1}{\sqrt{n}}$ is greater than $s-2$. Since, on A_n^ε,

$$p_j^{(m)}(\lambda) = h_{j+2}(\lambda)\, O(n^{\varepsilon(j+2)})$$

with the functions $h_k(\lambda)$ as already introduced in connection with (10), it is easy to see that

$$\sum_{m=1}^{s} \sum_{j=2m-2}^{sm-2} \left(\frac{1}{\sqrt{n}}\right)^j p_j^{(m)} =$$

$$= \sum_{l=1}^{s-2} \left(\frac{1}{\sqrt{n}}\right)^l \eta_l(\lambda) + \sum_{k=2}^{s} \sum_{l=(k-1)s-1}^{ks-2} \left(\frac{1}{\sqrt{n}}\right)^l (p_l^{(k)} + \ldots + p_l^{(\min([(l+2)/2], s))}) =$$

$$= \sum_{l=1}^{s-2} \left(\frac{1}{\sqrt{n}}\right)^l \eta_l(\lambda) + h_{s-2}^4(\lambda)\, O(n^{(\varepsilon-1/2)(s-1)+2\varepsilon}),$$

where

$$\eta_l(\lambda) = \sum_{m=1}^{[(l+2)/2]} p_l^{(m)}(\lambda) \tag{14}$$

and

$$|h_{s-2}^4(\lambda)| \leqq |\lambda|^{s^2/2}.$$

So we can continue our row (11) of equations the following way

$$= \mathbf{E} \exp\left\{-\lambda \int_0^1 \alpha^2(t)\, dt + \sum_{l=1}^{s-2} \left(\frac{1}{\sqrt{n}}\right)^l \eta_l(\lambda)\right\} \chi_{A_n^\varepsilon} + h_{s-2}^5(\lambda)\, O(n^{(\varepsilon-1/2)(s-1)+2\varepsilon}),$$

where

$$|h_{s-2}^5(\lambda)| \leqq |\lambda|^{s(s+2)/2}.$$

Putting now s instead of $s-2$, on the basis of (5), (6), (7) and (11), we have

$$\mathbf{E} \exp\{-\lambda\omega_n^2\} = \mathbf{E} \exp\left\{-\lambda \int_0^1 \alpha^2(t)\, dt + \sum_{l=1}^{s} \left(\frac{1}{\sqrt{n}}\right)^l \eta_l(\lambda)\right\} \chi_{A_n^\varepsilon} + \tag{15}$$

$$+ h_s^6(\lambda)\, O(n^{(\varepsilon-1/2)(s+1)+2\varepsilon}),$$

where

$$|h_s^6(\lambda)| \leqq |\lambda|^{(s+2)(s+4)/2}.$$

Considering the sum expression in the exponent of (15), we have, again by the MacLaurin formula,

$$\exp\left\{\sum_{l=1}^{s} \left(\frac{1}{\sqrt{n}}\right)^l \eta_l(\lambda)\right\} = 1 + \sum_{k=1}^{s} \left(\frac{1}{\sqrt{n}}\right)^k \zeta_k(\lambda) + R_{s+1}\left(\frac{1}{\sqrt{n}}\right), \tag{16}$$

where, via the Faa di Bruno formula (see Lemma 1 and formula (1.6) on page 169 in [25]) for the differentiation of compound functions, we have

$$(17)\qquad \zeta_k(\lambda)=\frac{1}{k!}\frac{d^k}{dx^k}\exp\left\{\sum_{l=1}^{s}x^l\eta_l(\lambda)\right\}\bigg|_{x=0}=\sum_{(i_1,\dots,i_k)}{}' \prod_{m=1}^{k}\frac{1}{i_m!}(\eta_m(\lambda))^{i_m}.$$

As to the Lagrange remainder term we get, again by the Faa di Bruno formula,

$$\left|R_{s+1}\left(\frac{1}{\sqrt{n}}\right)\right|=\left(\frac{1}{\sqrt{n}}\right)^{s+1}\left|\exp\left\{\sum_{l=1}^{s}\left(\frac{\vartheta}{\sqrt{n}}\right)^l\eta_l(\lambda)\right\}\right|\times$$
$$\times\left|\sum_{(k_1,\dots,k_{s+1})}{}' \prod_{m=1}^{s+1}\frac{1}{k_m!}\left\{\prod_{k=0}^{s-m}\binom{m+k}{m}\eta_{m+k}(\lambda)\left(\frac{\vartheta}{\sqrt{n}}\right)^k\right\}^{k_m}\right|,$$

where ϑ is a random variable with $0<\vartheta<1$. We also recall that $3\varepsilon-\frac{1}{2}<0$. Then, on A_n^ε, the exponential factor is majorized by

$$1+|\lambda|^{(s+2)/2}O(n^{3\varepsilon-1/2}),$$

while the last factor by

$$|\lambda|^{(s+2)(s+1)/2}O(n^{\varepsilon(s+1)+2\varepsilon(s+1)})$$

on applying (9) several times and noticing that on A_n^ε

$$\eta_l(\lambda)=(-\sqrt{-2\lambda})^{l+2}O(n^{\varepsilon(l+2)}).$$

Consequently, on A_n^ε,

$$(18)\qquad \left|R_{s+1}\left(\frac{1}{\sqrt{n}}\right)\right|\leqq|\lambda|^{(s+2)(s+2)/2}O(n^{(3\varepsilon-1/2)(s+1)}).$$

We get from (15) by (16) and (18) via (9) that

$$(19)\qquad \mathbf{E}\exp\{-\lambda\omega_n^2\}=\mathbf{E}\exp\left\{-\lambda\int_0^1\alpha^2(t)\,dt\right\}\left(1+\sum_{k=1}^{s}\left(\frac{1}{\sqrt{n}}\right)^k\zeta_k(\lambda)\right)+$$
$$+h_s(\lambda)O(n^{\delta-(s+1)/2}),$$

where $0<\delta=3\varepsilon(s+1)<\frac{1}{2}$, and for $h_s(\lambda)$ we already have

$$|h_s(\lambda)|\leqq|\lambda|^{(s+2)(s+4)/2}.$$

In these calculations we write $1+\chi_{A_n^\varepsilon}-1$ in place of the factor $\chi_{A_n^\varepsilon}$ in the expectation; then the new term with factor $\chi_{A_n^\varepsilon}-1$ decreases exponentially fast as $n\to\infty$. The latter can be shown exactly the same way as in (6).

From (1) and (19) it follows that

$$(20)\qquad \mathbf{E}\exp\left\{-\lambda\int_0^1\alpha^2(t)\,dt\right\}=\mathbf{E}\exp\{-\lambda\omega^2\}.$$

Let us also observe that if λ is real with $\lambda \geqq 0$, then

$$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \mathbf{E}\exp\left\{i\sqrt{2\lambda}\int_0^1 Y_n(x)\,dW(x)\right\}$$

is also real by the reflexivity of the Brownian Motion. From (19) and (20) it follows then that

(21) $$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \mathbf{E}\exp\{-\lambda\omega^2\} + \sum_{k=1}^{s}\left(\frac{1}{\sqrt{n}}\right)^k a_k^*(\lambda) + h_s(\lambda)\,O(n^{\delta-(s+1)/2})$$

where $a_k^*(\lambda) = \operatorname{Re} C_k(\lambda)$, and from (12), (13), (14), (17) and (19) we have

$$C_k(\lambda) = \sum_{(i_1,\dots,i_k)}' \frac{(-\sqrt{-2\lambda})^{k+2H_k}}{i_1!\dots i_k!}\mathbf{E}\exp\left\{-\lambda\int_0^1 \alpha^2(t)\,dt\right\}\Pi_{i_1,\dots,i_k},$$

where $\Pi i_{i_1,\dots,i_k}$ is already as in the formulation of the theorem with k in place of $2k$. If λ is real and nonnegative, then $a_k^*(\lambda)=0$, if k is odd, and if $k=2\nu$, $\nu=1,2,\dots$, then

$$a_{2\nu}^* = \sum_{(i_1,\dots,i_{2\nu})}' b_{i_1,\dots,i_{2\nu}}\lambda^{\nu+H_{2\nu}}\mathbf{E}\exp\left\{-\lambda\int_0^1 \alpha^2(t)\,dt\right\}\Pi_{i_1,\dots,i_{2\nu}}.$$

But a_k^* being an analytical function of λ, the same formulae hold true for any complex λ with $\operatorname{Re}\lambda \geqq 0$. Introducing $a_\nu(\lambda) = a_{2\nu}^*(\lambda)$, $\nu = 1, 2, \dots$, we can rewrite (21) the following way

$$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \mathbf{E}\exp\{-\lambda\omega^2\} + \sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k a_k(\lambda) + h_s(\lambda)\,O(n^{\delta-(s+1)/2})$$

valid for any complex λ with $\operatorname{Re}\lambda \geqq 0$. This was to be proved.

§ 4. **On the problem of inversion.** In the knowledge of the asymptotic expansion of Theorem 2 for the Laplace transform one should naturally like to invert it in order to get the corresponding form for the expanded distribution function. This task, unfortunately, is not accomplished here. A considerable work is done, however, towards this end. Our result is that the problem of the existence, of an (exactly computed) asymptotic expansion for $V_n(x) - V(x)$ is reduced to a qualitative problem concerning the behaviour of the characteristic function $f_n(t)$ of ω_n^2.

The following known results will be used.

Lemma A. (See e.g. [16]) *If the characteristic function $\tilde{f}(t)$ of an arbitrary distribution function $\tilde{F}(x)$ satisfies*

(22) $$\int_{-\infty}^{\infty} |t|^p |\tilde{f}(t)|\,dt < \infty,$$

with some integer $p \geqq 0$, then the $(p+1)^{st}$ derivative of $\tilde{F}(x)$ exists and

$$\tilde{F}^{(p+1)}(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Lemma B. (ESSEEN, see e.g. [25]) *Let $\tilde{F}(x)$ be a nondecreasing function and $\tilde{G}(x)$ a differentiable function of bounded variation on the real line, $\tilde{f}(t)$ and $\tilde{g}(t)$ the corresponding Fourier—Stieltjes transforms, $\tilde{F}(-\infty)=\tilde{G}(-\infty)$, $\tilde{F}(\infty)=\tilde{G}(\infty)$ and T an arbitrary positive number. Suppose $\sup_{-\infty<x<\infty} |\tilde{G}'(x)| \leqq C$ with some constant C. Then for any number $K_1 > \frac{1}{2\pi}$*

$$\sup_{-\infty<x<\infty} |\tilde{F}(x)-\tilde{G}(x)| \leqq K_1 \int_{-T}^{T} \left| \frac{\tilde{f}(t)-\tilde{g}(t)}{t} \right| dt + K_2 \frac{C}{T},$$

where K_2 is a positive constant depending only on K_1.

First we prove some lemmas needed in the sequel.

Lemma 1. *The distribution function $V(x)$ of the random variable ω^2 is arbitrary many times differentiable and for an arbitrary integer p*

$$V^{(p)}(x) \to 0 \quad as \quad |x| \to \infty.$$

Proof. For the characteristic function $f(t)$ of ω^2 (see (1)) we have, by direct computation

$$|f(t)| = \left| \left(\frac{\sqrt{-2it}}{\sinh \sqrt{-2it}} \right)^{1/2} \right| \leqq 2^{3/4} |t|^{1/4} \frac{\exp\left\{-\frac{1}{2}\sqrt{|t|}\right\}}{(1-\exp\{-2\sqrt{|t|}\})^{1/2}} \tag{23}$$

which shows that $f(t)$ satisfies condition (22) of Lemma A.

From inequality (23) we also have

Lemma 2. *For an arbitrary nonnegative real p*

$$|t|^p |f(t)| \to 0 \quad as \quad |t| \to \infty.$$

Now we show that our smooth distribution function $V(x)$ also rises smoothly from the point 0. Let (throughout the rest) $v(x) = \frac{d}{dx} V(x)$ be the density function of ω^2.

Lemma 3. *Denote (as before) by $v^{(q)}(x)$ the derivative of order q of $v(x)$. Then for an arbitrary q $(=0, 1, 2, \ldots,)$ we have*

$$v^{(q)}(0) = 0.$$

Proof. It vill be more comfortable to work now with the Laplace transform $\mathrm{E}e^{-\lambda\omega^2}$. By direct computation, again from $\left(\frac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}\right)^{1/2}$, we have

$$\int_0^\infty e^{-\lambda x}v(x)\,dx = 2^{3/4}\lambda^{1/4}\frac{\exp\left\{-\frac{1}{2}\sqrt{2\lambda}\right\}}{(1-\exp\{-2\sqrt{2\lambda}\})^{1/2}}.$$

Thus for real λ

$$\lim_{\lambda\to\infty}\lambda^q\int_0^\infty e^{-\lambda x}v(x)\,dx = 0 \quad (q=0,1,\dots).$$

Using this we get, by integration by parts

$$0=\lim_{\lambda\to\infty}\lambda\int_0^\infty e^{-\lambda x}v(x)\,dx = \int_0^\infty e^{-u}\left[\lim_{\lambda\to\infty} v\left(\frac{u}{\lambda}\right)\right]du = v(0),$$

$$0=\lim_{\lambda\to\infty}\lambda^2\int_0^\infty e^{-\lambda x}v(x)\,dx = \int_0^\infty e^{-u}u\left[\lim_{\lambda\to\infty}\frac{v(u/\lambda)}{u/\lambda}\right]du = v'(0),$$

$$0=\lim_{\lambda\to\infty}\lambda^3\int_0^\infty e^{-\lambda x}v(x)\,dx = \int_0^\infty e^{-u}u\left[\lim_{\lambda\to\infty}\frac{v'(u/\lambda)}{u/\lambda}\right]du = v''(0),$$

and so on. Hence the lemma is proved by induction.

It is very easy to see that $\alpha(t)=W(t)-\int_0^1 W(x)\,dx$ is a Gaussian process with $\mathrm{E}\alpha(t)=0$ and continuous covariance function $\min(s,t)-s\left(1-\frac{s}{2}\right)-t\left(1-\frac{t}{2}\right)+\frac{1}{3}$. Therefore it can be expanded in the following form (see [12])

$$\alpha(t)=\sum_{k=1}^\infty \xi_k\varphi_k(t), \tag{24}$$

where ξ_k $(k=1,2,\dots)$ is a normally distributed random variable with $\mathrm{E}\xi_k=0$, $\{\varphi_k\}_{k=1}^\infty$ is an orthonormal system of continuous functions on $[0,1]$ and the series in (24) converges with probability 1. Let us denote

$$\alpha_r=\int_0^1 \alpha^r(t)\,dt, \quad r=2,\dots,s. \tag{25}$$

The products of different powers of these α_r figure in the coefficients of the asymptotic expansion for the Laplace transform and we will need the following

Lemma 4. *For arbitrary $k_2,\dots,k_s\geqq 0$ the function* (of x) $\mathrm{E}\{\alpha_2^{k_2}\alpha_3^{k_3}\dots\alpha_s^{k_s}|\alpha_2=x\}$ *is differentiable as many times as we wish.*

Proof.
$$\mathbf{E}\{\alpha_2^{k_2}\alpha_3^{k_3}\dots\alpha_s^{k_s}|\alpha_2 = x\} =$$
$$= x^{k_2}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} x_3^{k_3}\dots x_s^{k_s}\,d\mathbf{P}\{\alpha_3 < x_3, \dots, \alpha_s < x_s|\alpha_2 = x\} =$$
$$= \frac{x^{k_2}}{v(x)}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} x_3^{k_3}\dots x_s^{k_s} f(x, x_3, \dots, x_s)\,dx_3\dots dx_s,$$
where $f(x_2, x_3, \dots, x_s)$ is the common density of the variables $\alpha_2, \alpha_3, \dots, \alpha_s$ (which will be shown to exist below). There we used that
$$\mathbf{P}\Big\{\int_0^1 \alpha^2(t)\,dt < x\Big\} = \mathbf{P}\{\omega^2 < x\} = V(x),$$
a consequence of relation (20) and the uniqueness theorem for Laplace transforms. Using Lemma 1, it is enough to show that $f(x_2, x_3, \dots, x_s)$ is arbitrary many times differentiable in x_2. Towards this end we will give an expression for this density.

Relation (24) gives

(26) $$\alpha_r = \sum_{k=1}^{\infty}\sum_{j_1+\dots+j_r=k} C_{j_1,\dots,j_r}\xi_{j_1}\cdot\ldots\cdot\xi_{j_r},$$

where
$$C_{j_1,\dots,j_r} = \int_0^1 \varphi_{j_1}(t)\cdot\ldots\cdot\varphi_{j_r}(t)\,dt.$$

Specifically, via the orthonormality of the φ_k system,

(27) $$\alpha_2 = \sum_{k=1}^{\infty}\xi_k^2,$$

with probability 1. We would like to get rid of the difficulty that in (26) an infinite number of Gaussian variables express α_r. Therefore we rewrite this expression the following way

(28) $$\alpha_r = \gamma^{r,0} + \sum_{i=1}^{s-1}\gamma_i^{r,1}\xi_i + \sum_{i,j=1}^{s-1}\gamma_{i,j}^{r,2}\xi_i\xi_j + \dots + \sum_{i_1,\dots,i_r=1}^{s-1}\gamma_{i_1,\dots,i_r}^{r,r}\xi_{i_1}\cdot\ldots\cdot\xi_{i_r},$$

$r=2, \dots, s$, where
$$\vec{\gamma}^{(r)} = \{\gamma^{r,0};\ \gamma_i^{r,1},\ i = 1, \dots, s-1;\ \dots;\ \gamma_{i_1,\dots,i_r}^{r,r},\ i_1, \dots, i_r = 1, \dots, s-1\}$$
are random variables depending on $\xi_s, \xi_{s+1}, \dots$, but not on $\xi_1, \dots, \xi_{s-1}$. Specifically, on the basis of (27) we have

(29) $$\alpha_2 = \gamma^{2,0} + \sum_{k=1}^{s-1}\xi_k^2.$$

From the introduced random vectors $\vec{\gamma}^{(r)}$, having $Q_r = \sum_{j=0}^{r} (s-1)^j$ components, formulate the random vector $\vec{\gamma} = (\vec{\gamma}^{(2)}, \ldots, \vec{\gamma}^{(r)})$, having $Q = \sum_{r=2}^{s} Q_r$ components. For each $r = 2, \ldots, s$, let $\overrightarrow{h^{(r)}}$ be the same way indexed nonrandom real vector as $\overrightarrow{\gamma^{(r)}}$, having Q_r components and let $\vec{h} = (\overrightarrow{h^{(2)}}, \ldots, \overrightarrow{h^{(r)}})$ be the corresponding Q component real vector. Let us consider the following system of algebraic equations

$$a_r = g_r(y_1, \ldots, y_{s-1}) \quad (r = 2, 3, \ldots, s), \tag{30}$$

where $a_2, a_3, \ldots, a_s$ are arbitrarily fixed real numbers. Further (from (29))

$$g_2(y_1, \ldots, y_{s-1}) = h^{2,0} + \sum_{i=1}^{s-1} y_i^2,$$

and, for $3 \leqq r \leqq s$, $g_r(y_1, \ldots, y_{s-1})$ is the right hand side of (28), having written h's and y's respectively, in place of γ's and ξ's. It is clear that the number of such vectors $\vec{h}$ for which the system (30) has infinitely many solutions $(y_1, \ldots, y_{s-1})$ is finite. Similarly, the Jacobian

$$J(y_1, \ldots, y_{s-1}) = \frac{\partial (g_2, \ldots, g_s)}{\partial (y_1, \ldots, y_{s-1})} \tag{31}$$

can be equal to zero only on hypersurfaces of the $s-1$ dimensional Euclidean space $\mathbf{R}^{s-1}$, which are defined by different vectors $\vec{h}$, and the number of such vectors is also finite. Let $\vec{\gamma} = \vec{h}$ be fixed, so that the Jacobian (31) is not zero and the system (30) has only a finite number of solution vectors $(y_1, \ldots, y_{s-1})$. This latter number we denote by q. Divide $\mathbf{R}^{s-1}$ onto q subspaces $U_1, \ldots, U_q$, so that in the interior of each one of them the system (30) would have only one solution. Denote by G the transformation $(y_1, \ldots, y_{s-1}) \to (x_2, \ldots, x_s)$ of $\mathbf{R}^{s-1}$ onto itself, defined by

$$g_r(y_1, \ldots, y_{s-1}) = x_r \quad (r = 2, 3, \ldots, s), \tag{32}$$

and let us define the functions $g_{2,k}^{-1}, \ldots, g_{s,k}^{-1}$, on the images $G(U_k)$, $k = 1, \ldots, q$, satisfying

$$g_{r,k}^{-1}(x_2, \ldots, x_s) = y_{r-1} \quad (r = 2, 3, \ldots, s),$$

if (32) holds.

Let

$$p(y_1, \ldots, y_{s-1}) = \prod_{k=1}^{s-1} \frac{1}{\sqrt{2\pi}\,\sigma_k} \exp\left\{\frac{-y_k^2}{2\sigma_k^2}\right\}$$

denote the common density of the independent normal variables $\xi_1, \ldots, \xi_{s-1}$. Then, after dividing the domain of integration onto intersections with the U_k's, changing the variables on these intersections and substituting the corresponding Jacobian determinant with the reciprocal of its inverse, for the common distribution function of $\alpha_2, \ldots, \alpha_s$ given $\vec{\gamma}=\vec{h}$ we get the following form

$$F(a_2, \ldots, a_s|\vec{\gamma} = \vec{h}) = \int\limits_{\substack{g_r(y_1, \ldots, y_{s-1})<a_r \\ r=2, \ldots, s}} \cdots \int p(y_1, \ldots, y_{s-1})\, dy_1 \ldots dy_{s-1} =$$

$$= \int\limits_{\substack{z_r<a_r \\ r=2, \ldots, s}} \cdots \int \sum_{k=1}^{q} \chi_{U_k}(z_2, \ldots, z_s) \frac{p(g_{2,k}^{-1}(z_2, \ldots, z_s), \ldots, g_{s,k}^{-1}(z_2, \ldots, z_s))}{|J(g_{2,k}^{-1}(z_2, \ldots, z_s), \ldots, g_{s,k}^{-1}(z_2, \ldots, z_s))|}\, dz_2 \ldots dz_s,$$

where χ_{U_k} is the indicator of the set U_k. This means that the conditional common density of the variables $\alpha_2, \ldots, \alpha_s$ given $\vec{\gamma}=\vec{h}$ is

$$f(a_2, \ldots, a_s|\vec{h}) = \sum_{\substack{(y_1, \ldots, y_{s-1}) \\ g_r(y_1, \ldots, y_{s-1})=a_r \\ r=2, \ldots, s}} \frac{p(y_1, \ldots, y_{s-1})}{|J(y_1, \ldots, y_{s-1})|}, \tag{33}$$

where the summation extends over all solutions $(y_1, \ldots, y_{s-1})$ of (30), i.e. the sum consists of q terms. Hence the common density of $\alpha_2, \ldots, \alpha_s$, which we were to find, has the form

$$f(a_2, \ldots, a_s) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{Q \text{ times}} f(a_2, \ldots, a_s|\vec{h})\, d\mathbf{P}\{\vec{\gamma} < \vec{h}\}.$$

It follows that for proving the arbitrarily many times differentiability of $f(a_2, \ldots, a_s)$ in a_2, it is enough to prove this for $f(a_2, \ldots, a_s|\vec{h})$.

Introduce polar coordinates

$$\begin{aligned} y_1 &= r\varkappa_1(\theta) = r\cos\theta_1 \\ y_2 &= r\varkappa_2(\theta) = r\sin\theta_1\cos\theta_2 \\ &\vdots \\ y_{s-1} &= r\varkappa_{s-1}(\theta) = r\sin\theta_1\sin\theta_2 \ldots \sin\theta_{s-3}\sin\theta_{s-2}, \end{aligned}$$

then

$$f(a_2, \ldots, a_s|\vec{h}) = \sum_{(r, \theta_1, \ldots, \theta_{s-2})} \frac{p(r\varkappa_1(\theta), \ldots, r\varkappa_{s-1}(\theta))}{|J(r\varkappa_1(\theta), \ldots, r\varkappa_{s-1}(\theta))|},$$

where the summation extends over all solutions $(r, \theta_1, \ldots, \theta_{s-2})$ of the following system of equations

$$\begin{aligned} a_2 &= h^{2,0} + r^2 \\ a_3 &= h^{3,0} + l_1^{(3)}(\theta)r + l_2^{(3)}(\theta)r^2 + l_3^{(3)}(\theta)r^3 \\ &\vdots \\ a_s &= h^{s,0} + l_1^{(s)}(\theta)r + \ldots + l_s^{(s)}(\theta)r^s. \end{aligned}$$

Here $l_j^{(i)}(\theta)$, $j=1, \dots, i$; $i=3, \dots, s$, is a trigonometric polynomial of $\theta_1, \dots, \theta_{s-2}$, already not depending on r. The sum now also consists of q terms, therefore it is enough to show the differentiability of the single summands. But differentiability in a_2 is equivalent to differentiability in r^2, and every summand is differentiable in r^2 as many times as we wish. Lemma 4 is proved.

As the random variable $\Pi_{i_1, \dots, i_{2k}}$, figuring in the coefficients of the asymptotic expansion of Theorem 2, is a linear combination of variables of the form $\alpha_2^{k_2} \cdot \ldots \cdot \alpha_s^{k_s}$, this latter result implies

Lemma 5. *For any* $i_1, \dots, i_{2k}$ *the function* $\mathbf{E}\left\{\Pi_{i_1, \dots, i_{2k}} \middle| \int\limits_0^1 \alpha^2(t)\,dt = x\right\}$ *has derivatives of arbitrary order.*

The following equation will be useful, by means of which the coefficients will be inverted.

Lemma 6. *For an arbitrary natural number* q,

$$\lambda^q \int\limits_0^\infty e^{-\lambda x} \mathbf{E}\{\Pi_{i_1, \dots, i_{2k}} | \alpha_2 = x\} v(x)\,dx = \int\limits_0^\infty e^{-\lambda x} \frac{d^q}{dx^q} [\mathbf{E}\{\Pi_{i_1, \dots, i_{2k}} | \alpha_2 = x\} v(x)]\,dx.$$

Proof. A row of q successive integrations by parts, where, at the k-th step we integrate the function $(-1)^{q-k} \dfrac{d^{q-k}}{dx^{q-k}} e^{-\lambda x}$ and differentiate the function $\varphi^{(k)}(x) = \dfrac{d^k}{dx^k} [\mathbf{E}\{\Pi_{i_2, \dots, i_{2k}} | \alpha_2 = x\} v(x)]$, $k=0, 1, \dots, q-1$. All the integrated out terms disappear, as by Lemma 3 we have $\varphi^{(k)}(0)=0$ for each k.

Since, by this Lemma 6, we have

$$\lambda^q \mathbf{E} \exp\{-\lambda \alpha_2\} \Pi_{i_1, \dots, i_{2k}} =$$

$$= \lambda^q \int\limits_0^\infty e^{-\lambda x} \mathbf{E}\{\Pi_{i_1, \dots, i_{2k}} | \alpha_2 = x\} v(x)\,dx =$$

$$= -\frac{1}{\lambda} \int\limits_0^\infty e^{-\lambda x} \frac{d^{q+1}}{dx^{q+1}} [\mathbf{E}\{\Pi_{i_1, \dots, i_{2k}} | \alpha_2 = x\} v(x)]\,dx,$$

we also proved the following

Lemma 7. *For any natural* q *and* $i_1, \dots, i_2 \geqq 0$ *we have, as* $|\lambda| \to \infty$,

$$|\lambda|^q \left| \mathbf{E} \exp\left\{-\lambda \int\limits_0^1 \alpha^2(t)\,dt\right\} \Pi_{i_1, \dots, i_{2k}} \right| \to 0.$$

We now start inverting the asymptotic expansion of Theorem 2. Integrating by parts we have

$$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \lambda\int_0^\infty e^{-\lambda x}V_n(x)\,dx$$

and

$$\mathbf{E}\exp\{-\lambda\omega^2\} = \lambda\int_0^\infty e^{-\lambda x}V(x)\,dx.$$

This implies

$$\int_0^\infty e^{-\lambda x}V_n(x)\,dx = \int_0^\infty e^{-\lambda x}V(x)\,dx + \sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k\frac{a_k(\lambda)}{\lambda} + \frac{h_s(\lambda)}{\lambda}O(n^{\varepsilon-(s+1)/2}),$$

and, by Lemma 6, we have

$$\int_0^\infty e^{-\lambda x}V_n(x)\,dx = \int_0^\infty e^{-\lambda x}\left\{V(x) + \sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k\psi_k(x)\right\}dx + \frac{h_s(\lambda)}{\lambda}O(n^{\varepsilon-(s+1)/2})$$

where

$$\psi_k(x) = \sum_{(i_1,\dots,i_{2k})}{}' b_{i_1,\dots,i_{2k}}\frac{d^{k-1+H_{2k}}}{dx^{k-1+H_{2k}}}[\mathbf{E}\{\Pi_{i_1,\dots,i_{2k}}|\alpha_2 = x\}v(x)].$$

The functions $V_n(x)$, $V(x)$ and $\psi_k(x)$, $k=1,\dots,\left[\frac{s}{2}\right]$, are continuous and are of bounded variation on each finite interval from $[0,\infty)$; further, the Laplace transform itself here has abscissa of convergence 0. Therefore, the complex inversion formula can be applied for the left hand side, and also for the first term of the right hand side. We write formally

$$V_n(x) = V(x) + \sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k\psi_k(x) + A(\varepsilon, s, n, x), \tag{34}$$

where $A=A(\varepsilon, s, n, x)$ is a function, into which $\dfrac{h_s(\lambda)}{\lambda}O(n^{\varepsilon-(s+1)/2})$ is inverted.

For justification of an asymptotic expansion of type (34). i.e. to estimate the remainder term A here, usually Essen's result of Lemma B is applied. For doing this we rewrite Theorem 2 in terms of characteristic functions.

$$\mathbf{E}e^{it\omega_n^2} = \mathbf{E}e^{it\omega^2} + \sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k a_k(-it) + h_s(-it)O(n^{\varepsilon-(s+1)/2}).$$

Put

$$f_n(t) = \int_0^\infty e^{itx}\,dV_n(x) = \mathbf{E}e^{it\omega_n^2},$$

and define the functions $G_{n,s}$ by the following equation

$$g_{n,s}(t) = \int_{-\infty}^\infty e^{itx}\,dG_{n,s}(x) = \mathbf{E}e^{it\omega^2} + \sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k a_k(-it).$$

This means

$$G_{n,s}(x) = V(x) + \sum_{k=1}^{[s/2]} \left(\frac{1}{n}\right)^k \psi_k(x).$$

It is easy to see that $V_n(x)$ and $G_{n,s}(x)$ satisfy the conditions of Lemma B. Specifically, from Lemmas 1 and 5 it follows that $G_{n,s}$ is of bounded variation and, from the existence of the integrals of Lemma 6, that $\psi_k(0)=\psi_k(\infty)=0$; that is $V_n(-\infty)= =G_{n,s}(-\infty)=0$ and $V_n(\infty)=G_{n,s}(\infty)=1$. Put $C=\sup |G'_{n,s}(x)|$. Then, by Lemma B, we have

$$\sup_{-\infty<x<\infty} |V_n(x)-G_{n,s}(x)| = \sup_{-\infty<x<\infty} |A(\varepsilon, s, n, x)| \leqq$$

$$\leqq K_2 \frac{C}{T} + K_1 \int_{-T}^{T} \left| \frac{f_n(t)-g_{n,s}(t)}{t} \right| dt.$$

Put $T=n^\beta$ where $\beta=\dfrac{s+1}{2}-2\varepsilon$ and $\delta=\dfrac{2\varepsilon}{(s+2)(s+4)}$. Then

$$\int_{-n^\beta}^{n^\beta} \left| \frac{f_n(t)-g_{n,s}(t)}{t} \right| dt \leqq \int_{-n^\delta}^{n^\delta} \left| \frac{f_n(t)-g_{n,s}(t)}{t} \right| dt +$$

$$+ \int_{n^\delta \leqq |t| \leqq n^\beta} \left| \frac{g_{n,s}(t)}{t} \right| dt + \int_{n^\delta \leqq |t| \leqq n^\beta} \left| \frac{f_n(t)}{t} \right| dt = I_1 + I_2 + I_3,$$

where for the first term we have by Theorem 2

$$I_1 = O(n^{\varepsilon-(s+1)/2}) \int_{-n^\delta}^{n^\delta} \left| \frac{h_s(-it)}{t} \right| dt =$$

$$= O(n^{\varepsilon-(s+1)/2}) \Big(\int_{-1}^{1} |t|^{-1/2} dt + \int_{1 \leqq |t| \leqq n^\delta} |t|^{(s+2)(s+4)/2-1} dt \Big) =$$

$$= O(n^{\varepsilon-(s+1)/2+\delta(s+2)(s+4)/2}) = O(n^{2\varepsilon-(s+1)/2}).$$

For estimating the second one, let us observe that

$$|g_{n,s}(t)| \leqq |f(t)| + \sum_{k=1}^{[s/2]} \left(\frac{1}{n}\right)^k |a_k(-it)|,$$

and that $|a_k(-it)|$ is majorized by a linear combination of functions of the form

$$|t|^{k+H_{2k}} |\mathbf{E} e^{it\alpha_2} \Pi_{i_1,\dots,i_{2k}}|.$$

Therefore, by Lemmas 2 and 7, there exists (for any positive number m) a constant C_m such that

$$|g_{n,s}(t)| \leqq \frac{C_m}{|t|^m}.$$

This implies

$$I_2 \leqq \int_{n^\delta \leqq |t| < \infty} C_m |t|^{-(m+1)}\, dt = O(n^{-m\delta}) = O(n^{2\varepsilon-(s+1)/2})$$

as m was arbitrary. Unfortunately, we do not have any estimate for I_3. On this way there exists a constant K_s depending only on s, so that

$$\sup_{-\infty<x<\infty} |A(\varepsilon, s, n, x)| \leqq K_s n^{2\varepsilon-(s+1)/2} + K_1 \int_{T_n(s, 2\varepsilon)} \left|\frac{f_n(t)}{t}\right| dt, \tag{35}$$

where

$$T_n(s, \varepsilon) = \{t : n^{\varepsilon/(s+2)(s+4)} \leqq |t| \leqq n^{(s+1)/2-\varepsilon}\}. \tag{36}$$

By (34) and (35) we then have

Theorem 3. *For any natural s and real positive* ε

$$V_n(x) - V(x) = \sum_{k=1}^{[s/2]} \left(\frac{1}{n}\right)^k \psi_k(x) + O(n^{-(s+1)/2+\varepsilon}) + O\left(\int_{T_n(s,\varepsilon)} \left|\frac{f_n(t)}{t}\right| dt\right),$$

where

$$\psi_k(x) = \sum_{(i_1, \dots, i_{2k})}' b_{i_1, \dots, i_{2k}} \frac{d^{k-1+H_{2k}}}{dx^{k-1+H_{2k}}} \left[\mathrm{E}\left\{\Pi_{i_1, \dots, i_{2k}} \Big| \int_0^1 \alpha^2(t)\, dt\right\} v(x)\right],$$

$\alpha(t)$, $b_{i_1, \dots, i_{2k}}$, H_{2k} *and* $\Pi_{i_1, \dots, i_{2k}}$ *are as in Theorem 2,* $v(x)$ *is the density of* ω^2, $f_n(t)$ *is the characteristic function of* ω_n^2 *and* $T_n(s, \varepsilon)$ *is as in* (36).

§ 5. **Remarks, conjectures.** Now by Theorem 3 the existence of an asymptotic expansion $\left(\text{surprisingly according to powers of } \frac{1}{n} \text{ instead of those of } \frac{1}{\sqrt{n}}\right)$ is reduced to the behaviour of $f_n(t)$. In this connection it is natural to make the following

Conjecture: $\displaystyle\int_{T_n(s,\varepsilon)} \left|\frac{f_n(t)}{t}\right| dt = O(n^{\varepsilon-(s+1)/2})$, *i.e. the asymptotic expansion*

$$V_n(x) = V(x) + \sum_{k=1}^{[s/2]} \left(\frac{1}{n}\right)^k \psi_k(x) + O(n^{-(s+1)/2+\varepsilon})$$

holds true.

For this, of course, it would be enough to prove that $|f_n(t)|$ decreases faster than any power of $|t|$, as $|t| \to \infty$, just like the limiting characteristic function (Lemma 2). Or, equivalently, it would be enough to prove that the sequence $f_n(t)$ of our characteristic functions converges uniformly *on the whole real line* to $f(t)$. As a matter of fact it would be enough to show that $|t|^S |f_n(t)| \to 0$, as $|t| \to \infty$, where $S = \left(\frac{s+1}{2} - \varepsilon\right) \frac{(s+2)(s+4)}{\varepsilon}$, or, equivalently, that $V_n(x)$ is $(S+1)$-times differentiable.

In the special case $s=2$, which would be important in practical applications, the coefficient $\psi_1(x)$ can easily be computed. One gets

(37)
$$V_n(x)-V(x)=\frac{1}{n}\left(-\frac{1}{2}x\,v(x)-\frac{1}{4}x^2v'(x)\right)+O(n^{-3/2+\varepsilon})+O\left(\int\limits_{T_n(2,\varepsilon)}\left|\frac{f_n(t)}{t}\right|dt\right),$$

where, specifically,

$$T_n(2,\varepsilon)=\{t\colon n^{\varepsilon/24}\leqq|t|\leqq n^{3/2-\varepsilon}\}.$$

It should be remarked that (37) can be proved without our general Lemmas 4, 5, 6 and 7 because, here, $a_1(\lambda)$ has the following simple form

$$a_1(\lambda)=-\frac{\lambda^2}{2}\mathbf{E}e^{-\lambda\alpha_2}\alpha_2^2,$$

and for instance we get consequently

$$g_{n,2}(t)=-\frac{t^2}{n}f''(t),\quad\text{where}\quad f(t)=\left(\frac{\sqrt{-2it}}{\sinh\sqrt{-2it}}\right)^{1/2},$$

and the corresponding estimates can be computed in a direct way.

In addition we prove the following simple fact.

Lemma 8. *For any real $p\geqq 0$ and integers $q=0,1,2,\ldots$ the function $x^pv^{(q)}(x)$ is bounded on $(-\infty,\infty)$.*

Proof. It is enough to show this for $p>0$. From the inversion formula for Fourier transforms we have

$$v^{(q)}(x)=(-i)^q\int\limits_{-\infty}^{\infty}e^{-itx}t^qf(t)\,dt.$$

Integrating p times by parts

$$x^pv^{(q)}(x)=(-i)^{q+p}\int\limits_{-\infty}^{\infty}e^{-itx}\frac{d^p}{dt^p}(t^qf(t))\,dt,$$

whence

$$|x^pv^{(q)}(x)|\leqq\int\limits_{-\infty}^{\infty}\left|\frac{d^p}{dt^p}\left\{t^q\frac{\sqrt{-2it}}{\sinh\sqrt{-2it}}\right\}\right|dt<\infty.$$

From the special case $s=2$ of our Conjecture and this Lemma 8 we would have the rate of convergence

$$\Delta_n=\sup_{-\infty<x<\infty}|V_n(x)-V(x)|=O\left(\frac{1}{n}\right).$$

For the latter it would be enough to show only that, if $|t|\to\infty$, then $|t|^{48}|f_n(t)|\to 0$.

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